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BESSEL PROCESSES, STOCHASTIC VOLATILITY, AND TIMER OPTIONS

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Motivated by analytical valuation of timer options (an important innovation in realized variance-based derivatives), we explore their novel mathematical connection with stochastic volatility and Bessel processes (with constant drift). Under the Heston (1993) stochastic volatility model, we formulate the problem through a first-passage time problem on realized variance, and generalize the standard risk-neutral valuation theory for fixed maturity options to a case involving random maturity. By time change and the general theory of Markov diffusions, we characterize the joint distribution of the first-passage time of the realized variance and the corresponding variance using Bessel processes with drift. Thus, explicit formulas for a useful joint density related to Bessel processes are derived via Laplace transform inversion. Based on these theoretical findings, we obtain a Black–Scholes–Merton-type formula for pricing timer options, and thus extend the analytical tractability of the Heston model. Several issues regarding the numerical implementation are briefly discussed.

KEY WORDS: timer options, volatility derivatives, realized variance, stochastic volatility models, Bessel processes.

1. INTRODUCTION

Over the past decades, volatility has become one of the central issues in financial modeling. Both the historic volatility derived from time series of past market prices and the implied volatility derived from the market price of a traded derivative (in particular, an option) play important roles in derivatives valuation. In addition to index or stock options, a variety of volatility (or variance) derivatives, such as variance swaps, volatility swaps, and options on VIX (the Chicago board of exchange volatility index), are now actively traded in the financial security markets.

As a financial innovation, Société Générale Corporate and Investment Banking (SG CIB) launched a new type of option (see Sawyer 2007; Fontenay 2007; or Hawkins and Krol 2008), called "timer option,"for managing volatility risk. As reported in Sawyer (2007), "the price of a vanilla call option is determined by the level of implied volatility quoted in the market, as well as maturity and strike price. But the level of implied volatility is often higher than realized volatility, reflecting the uncertainty of future market direction. In simple terms, buyers of vanilla calls often overpay for their options. In fact,

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having analyzed all stocks in the Euro Stoxx 50 index since 2000, SG CIB calculates that 80% of 3-month calls that have matured in-the-money were overpriced." To ensure that investors pay for the realized variance instead of the implied one, a timer call (put) option entitles the investor to the right to purchase (sell) the underlying asset at a prespecified strike price at the first time when a prespecified variance budget is consumed. Instead of fixing the maturity and letting the volatility float, timer options do the reverse by fixing the variance budget and randomizing the maturity. Thus, a timer option can be viewed as a call option with random maturity, where the maturity occurs at the first time a prescribed variance budget is exhausted.

Several features of timer options can be seen as follows. First, according to Société Générale, a timer call option is empirically cheaper than a traditional European call option with the same expected investment horizon, when the realized volatility is less than the implied volatility. Second, with timer options, systematic market timing is optimized for the following reason. If the volatility increases, the timer call option terminates earlier. However, if the volatility decreases, the timer call option simply takes more time to reach its maturity. Third, financial institutions can use timer options to overcome the difficulty of pricing the call and put options whose implied volatility is difficult to quote. This situation usually happens in the markets where the implied volatility data does not exist or is limited. Fourth, in consideration of applications to portfolio insurance, portfolio managers can use timer put options on an index (or a well-diversified portfolio) to limit their downside risk. They might be interested in hedging specifically against sudden market drops such as the crashes in 1987 and 2008. Fifth, from the perspective of the financial institutions who offer timer options, if there is a market collapse, the sudden high volatility will cause the timer put options to be exercised rapidly, thus protecting and hedging the fund's value. By contrast, European put options do not have this feature. With a timer put option, some uncertainty about the portfolio's outcome is represented by uncertainty about the variable time horizon (see Bick 1995 for a similar discussion).

Stochastic volatility models have widely been employed in option valuation, see, e.g., surveys in Broadie and Detemple (2004) and Fouque, Papanicolaou, and Sircar (2000). In particular, applications in pricing volatility derivatives can be found in, e.g., Detemple and Osakwe (2000), Broadie and Jain (2008), and Kallsen, Muhle-Karbe, and Voß (2011) among others. The motivation of this paper lies in the modeling and valuation for timer options under the celebrated Heston stochastic volatility model (see Heston 1993), which is popular among others for its analytical tractability. To generalize the risk-neutral valuation for options with fixed maturities to random maturities, we begin by formulating the timer option valuation problem as a first-passage time problem. Based on such a representation, a conditional Black–Scholes–Merton-type formula for pricing timer options follows immediately. To derive explicit formulas for pricing timer options, we apply stochastic time-change techniques to find that the variance process in the Heston model, running on first-passage time of the realized variance, is indeed equivalent in distribution to a Bessel process with constant drift (see Linetsky 2004), which is known for its application in queuing theory and financial engineering. This naturally motivates the investigation of analytical formulas for a joint density on Bessel processes and the integration of its reciprocal via Laplace transform inversion. Finally, explicit Black–Scholes–Merton-type formulas for pricing timer options are established as novel generalizations of Black and Scholes (1973), Merton (1973), and Heston (1993). The challenge of practical implementation for the analytical formula, as demonstrated in this paper, can be circumvented by an application of the algorithm proposed by Abate and Whitt (1992) for efficient computation of Laplace transform inversion via Fourier series expansion.

In the Heston (1993) stochastic volatility model, a Feller square root diffusion (see Feller 1951; also known for its application in modeling spot interest rates, see Cox, Ingersoll, and Ross 1985) is employed for modeling the stochastic variance process. As one of the most popular and widely used stochastic volatility models, the variance process $\{V_t\}$ is assumed to follow the stochastic differential equation:

$$dV_t = \kappa(\theta - V_t)\,dt + \sigma_v\sqrt{V_t}dW_t,$$

where $\{W_t\}$ is a standard Brownian motion. The Feller diffusion (see Feller 1951), also called "square root diffusion," is widely used in financial modeling due to its favorable properties. The Heston (1993) stochastic volatility model has received much attention from both academia and industry due to its analytical tractability. The popularity started from its explicit formulas for pricing European options via the inversion of closed-form characteristic functions proposed in Heston (1993). We also note that exact simulation strategies based on the explicit distributions of the Heston model have been proposed, see, e.g., Broadie and Kaya (2006) and Glasserman and Kim (2011). However, the distribution of first-passage time of realized variance, as advocated for converting variance uncertainty to time randomness in Geman and Yor (1993), has not yet drawn enough attention. Mathematically speaking, the first-passage time for realized variance can be defined through the first time for the total realized variance to achieve a certain level, i.e.,

$$\tau_b := \inf\left\{u \geq 0;\ \int_0^u V_s ds = b\right\}, \tag{1.1}$$

for any $b > 0$. It is obvious that such a random time runs fast if the volatility is high and runs slow if the volatility is low. For the Hull and White (1987) stochastic volatility model, Geman and Yor (1993) established an explicit formula for the distribution related to τ_b using some remarkable analytical properties of Bessel processes. In this paper, motivated by the pricing of timer options, we connect the distribution of (τ_b, V_{τ_b}) to Bessel processes via stochastic time change and the general theory of Markov processes; see Revuz and Yor (1999).

The organization of the rest of this paper is as follows. In Section 2 we introduce the model and some basic setup. In Section 3 we formulate the timer option valuation problem as a first-passage time problem. In Section 4 we investigate the connection between the Feller square root diffusion and Bessel process with constant drift and derive a joint density of Bessel processes needed for characterizing the distribution of our interest. In Section 5 Black–Scholes–Merton-type formulas for pricing timer options are proposed and analyzed in comparison to existing literature on business time hedging and quadratic-variation-based strategies. In Section 6, an efficient algorithm for implementing the formulas is proposed and demonstrated through numerical examples. Section 7 concludes this paper and points out some limitations and further research opportunities. All proofs are collected in Appendices A, B, and C.

2. BASIC SETUP AND THE MODEL

For an investment horizon T large enough, let us define $\Delta t = T/n$ as the time increment and suppose that the asset price is observed at $t_i = i\Delta t$ for $i = 0, 1, 2, \ldots, n$. For example, according to the daily sampling convention, Δt is usually chosen as $1/252$ corresponding to the standard 252 trading days in a year. Let $\{S_t\}$ denote the price process of the underlying stock (or index). The annualized realized variance for the period $[0, T]$ is defined as

$$\widehat{\sigma_T^2} := \frac{1}{(n-1)\Delta t}\sum_{i=0}^{n-1}\left(\log\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2,$$

see, e.g., Broadie and Jain (2008). The (cumulative) realized variance over time period $[0, T]$ is accordingly defined as

$$RV_T = T\cdot\widehat{\sigma_T^2} \approx \sum_{i=0}^{n-1}\left(\log\frac{S_{t_{i+1}}}{S_{t_i}}\right)^2, \tag{2.1}$$

which records the total variation of the asset return. Upon purchasing a timer call option, an investor specifies a variance budget calculated from $B = \sigma_0^2 T_0$, where T_0 is an expected investment horizon and σ_0 is the forecasted realized volatility during the expected investment period. Thus, a timer call option pays out $\max(S_{\mathcal{T}} - K, 0)$ at the first time $\mathcal{T}$ when the realized variance (2.1) exceeds the level B, i.e.,

$$\mathcal{T} := \min\left\{k, \sum_{i=1}^{k}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 \geq B\right\}\cdot\Delta t. \tag{2.2}$$

Similarly, a timer put option with strike K and variance budget B has a payoff $\max(K - S_{\mathcal{T}}, 0)$.

In this paper, we assume that the asset $\{S_t\}$ and its instantaneous variance $\{V_t\}$ follow the Heston stochastic volatility model (see Heston 1993). In a filtered probability space $(\Omega, \mathbb{P}, \mathcal{G}, \{\mathcal{G}_t\})$, the joint dynamics of $\{S_t\}$ and $\{V_t\}$ are specified as

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{V_t}S_t\left(\rho dZ_t^{(1)} + \sqrt{1-\rho^2}dZ_t^{(2)}\right),\\ dV_t &= \epsilon(\vartheta - V_t)\,dt + \sigma_v\sqrt{V_t}dZ_t^{(1)}, \end{aligned}$$

where $\{(Z_t^{(1)}, Z_t^{(2)})\}$ is a standard two-dimensional Brownian motion. Here, μ represents the return of the asset; ϵ is the speed of mean reversion of $\{V_t\}$; ϑ is the long-term mean-reversion level of $\{V_t\}$; σ_v is a parameter reflecting the volatility of $\{V_t\}$; ρ is the correlation between the asset return and its variance. Let us also recall that the Heston stochastic volatility model is equipped with a particular linear functional form of the market price of volatility risk $\Lambda(t, V_t) = \eta\sqrt{V_t}$.

For computational convenience, the valuation of variance (volatility) derivatives usually calls for continuous approximation of the realized variance (2.1). Through quadratic variation calculation for $\{\log S(t)\}$, it is straightforward to find that, for any $t > 0$,

$$\lim_{\Delta t\to 0}\sum_{i=1}^{t/\Delta t}\left(\log\frac{S_{t_i}}{S_{t_{i-1}}}\right)^2 = \lim_{\Delta t\to 0}\sum_{i=1}^{t/\Delta t}\left(\log S_{t_i} - \log S_{t_{i-1}}\right)^2 = \int_0^t V_s ds.$$

Thus, we define

$$I_t := \int_0^t V_s ds \tag{2.3}$$

as a continuous-time version of the cumulative-realized variance (2.1) over the time period $[0, t]$. As a special case of the τ_b defined in (1.1), we introduce a first-passage time

$$\tau := \inf\left\{u \geq 0; \int_0^u V_s ds = B\right\}. \tag{2.4}$$

It is obvious that τ is a continuous-time approximation of $\mathcal{T}$ defined in (2.2). In a financial market with relatively high trading frequency, e.g., daily, a timer call option can be regarded as an option paying out $\max(S_\tau - K, 0)$ at the random maturity time (2.4). In the following expositions, we will focus on this continuous-time setting.

3. TIMER OPTION VALUATION AS A FIRST-PASSAGE TIME PROBLEM

We assume that timer options can be dynamically replicated using the underlying and auxiliary assets reflecting market price of volatility risk. Without loss of generality, we employ variance swaps to do such a job. Because the replication needs to be done until the random maturity τ, we replace expired variance swaps with new ones until the total variance budget is consumed at the time τ. More precisely, we regard $[0, \tau]$ as a disjoint union of hedging periods

$$D_i := \left[\left(\sum_{j=0}^{i-1} T_j\right) \wedge \tau, \left(\sum_{j=0}^{i} T_j\right) \wedge \tau\right] \quad \text{for } i = 1, 2, \ldots,$$

where $T_0 := 0$ and $T_1, T_2, \ldots$ represent the maturities of the variance swaps employed in replication. In other words, on each D_i, the timer option is replicated by dynamically rebalancing the portfolio consisting of the underlying asset with price $\{S_t\}$ and a variance swap with maturity T_i and price process $\{G_t(i)\}$.

By generalizing the risk-neutral valuation theory for pricing derivative securities with fixed maturity to a case with random maturity, a heuristic replication argument allows us to establish a boundary value problem for pricing timer options, dynamic hedging strategies, as well as risk-neutral expectation representations for timer option prices in what follows. The literature of stochastic volatility has witnessed various treatment of replication and valuation, see, e.g., Cvitanić, Pham, and Touzi (1999), Frey and Sin (1999), Frey (2000), Hobson (2004), Biagini, Guasoni, and Pratelli (2000), Hofmann, Platen, and Schweizer (1992), and Romano and Touzi (1997). However, in this paper, we assume that the market is completed by trading auxiliary volatility-sensitive derivatives, e.g., variance swaps, which are priced in a risk-neutral probability measure $\mathbb{Q}$. Under $\mathbb{Q}$, the Heston model dynamics follows

$$dS_t = rS_t dt + \sqrt{V_t} S_t \left[\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}\right], \tag{3.1a}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_v \sqrt{V_t} dW_t^{(1)}, \tag{3.1b}$$

for some $\kappa, \theta, \sigma_v > 0$. Here, r is the risk-free rate. We also assume that $-1 < \rho < 1$. Without loss of generality, we assume that a variance swap with large enough maturity struck at some level can be used as an auxiliary asset for replicating timer options. According to Broadie and Jain (2008), the price for variance swap $G_t = G(t, V_t, I_t)$ for some function G on $\mathbb{R}_+^3$ of the class $\mathcal{C}^{1,2,1}$ satisfies the PDE

$$\frac{\partial G}{\partial t} + \kappa(\theta - v)\frac{\partial G}{\partial v} + v\frac{\partial G}{\partial x} + \frac{1}{2}\sigma_v^2 v \frac{\partial^2 G}{\partial v^2} = rG,$$

and

$$d(e^{-rt}G_t) = e^{-rt}\frac{\partial G}{\partial v}\sigma_v\sqrt{V_t}dW_t^{(1)}.$$

To replicate a timer option, suppose at time t, an investor holds Δ_t^S shares of the underlying asset with price S_t and Δ_t^G shares of the aforementioned variance swap with price G_t. The remainder of the portfolio value, $\Pi_t - \Delta_t^S S_t - \Delta_t^G G_t$, is fully invested in the risk-free money market account. In order for the portfolio to be self-financing, we have

$$d\Pi_t = \Delta_t^S dS_t + \Delta_t^G dG_t + r\left(\Pi_t - \Delta_t^S S_t - \Delta_t^G G_t\right) dt,$$

which is equivalent to

$$\begin{aligned} d(e^{-rt}\Pi_t) &= e^{-rt}\left[\Delta_t^S(dS_t - rS_t dt) + \Delta_t^G(dG_t - rG_t dt)\right] \\ &= e^{-rt}\left[\Delta_t^G\frac{\partial G}{\partial v}\sigma_v\sqrt{V_t}dW_t^{(1)} + \Delta_t^S\sqrt{V_t}S_t\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right)\right]. \end{aligned}$$

On the other hand, we assume that the timer (put or call) option struck at K with payoff function $H(s)$, i.e.,

(3.2a) $$H(s) = \max\{K - s, 0\} \quad \text{for a timer put option,}$$

(3.2b) $$H(s) = \max\{s - K, 0\} \quad \text{for a timer call option,}$$

has price $P_{t\wedge\tau} = u(t\wedge\tau; S_{t\wedge\tau}, V_{t\wedge\tau}, I_{t\wedge\tau})$ for some function $u(t; s, v, x)$ on $\mathbb{R}_+^4$ of the class $\mathcal{C}^{1,2,2,1}$. In other words, $u(t, s, v, x)$ denotes the price of the timer option at time t with underlying value s, variance level v, strike K, and variance budget $B - x$. Thus, we have

$$\begin{aligned} d(e^{-rt}P_t) = e^{-rt}\Bigg[&\frac{\partial u}{\partial t} + \kappa(\theta - V_t)\frac{\partial u}{\partial v} + rS_t\frac{\partial u}{\partial s} + V_t\frac{\partial u}{\partial x} + \frac{1}{2}\sigma_v^2 V_t\frac{\partial^2 u}{\partial v^2} + \frac{1}{2}S_t^2 V_t\frac{\partial^2 u}{\partial s^2} \\ &+ \rho\sigma_v S_t V_t\frac{\partial^2 u}{\partial s\partial v} - ru\Bigg]dt + e^{-rt}\Bigg[\frac{\partial u}{\partial v}\sigma_v\sqrt{V_t}dW_t^{(1)} \\ &+ \frac{\partial u}{\partial s}\sqrt{V_t}S_t\left(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right)\Bigg]. \end{aligned}$$

Replication yields that

$$d(e^{-rt}P_t) = d(e^{-rt}\Pi_t),$$

which results in the following PDE boundary value problem

$$\frac{\partial u}{\partial t} + \kappa(\theta - v)\frac{\partial u}{\partial v} + rs\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial x} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 u}{\partial v^2} + \frac{1}{2}s^2 v\frac{\partial^2 u}{\partial s^2} + \rho\sigma_v sv\frac{\partial^2 u}{\partial s\partial v} - ru = 0,$$
(3.3a)

(3.3b) $$u(t, s, v, B) = H(s),$$

for $(t, s, v, x) \in [0, +\infty) \times [0, +\infty) \times (0, +\infty) \times (0, B]$ as well as the following replicating strategy:

$$\Delta_t^S = \frac{\partial u}{\partial s} \quad \text{and} \quad \Delta_t^G = \frac{\partial u}{\partial v} \Big/ \frac{\partial \mathcal{G}}{\partial v}.$$

In the following proposition, we represent the timer option price as a risk-neutral expectation of discounted payoff, which further leads to a conditional Black–Scholes–Merton-type formula. For such a purpose, we define the following functions, which generalize the corresponding components in the celebrated Black–Scholes–Merton formula (see Black and Scholes 1973 and Merton 1973) for pricing European options. We let

$$d_0(v, \xi) := \frac{\rho}{\sigma_v}(v - V_0 - \kappa\theta\xi + \kappa B) - \frac{1}{2}\rho^2 B, \tag{3.4a}$$

$$d_1(v, \xi) := \frac{1}{\sqrt{(1-\rho^2)B}}\left[\log\left(\frac{S_0}{K}\right) + r\xi + \frac{1}{2}B(1-\rho^2) + d_0(v, \xi)\right], \tag{3.4b}$$

$$d_2(v, \xi) := \frac{1}{\sqrt{(1-\rho^2)B}}\left[\log\left(\frac{S_0}{K}\right) + r\xi - \frac{1}{2}B(1-\rho^2) + d_0(v, \xi)\right], \tag{3.4c}$$

where, indeed, the following relation holds

$$d_2(v, \xi) = d_1(v, \xi) - \sqrt{(1-\rho^2)B}.$$

Also, denote by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

the standard normal cumulative distribution function.

PROPOSITION 3.1. *The price of a timer put or call option with variance budget B and payoff function $H(s)$ defined in (3.2a) or (3.2b) admits the following Feynman–Kac-type representation for its arbitrage-free price at time $t \wedge \tau$,*

$$u(t \wedge \tau, s, v, x) = \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau - t\wedge\tau)} H(S_\tau) \mid S_{t\wedge\tau} = s,\ V_{t\wedge\tau} = v,\ I_{t\wedge\tau} = x], \tag{3.5}$$

which is indeed independent of $t \wedge \tau$ in the sense that

$$u(t \wedge \tau, s, v, x) \equiv \mathbb{E}^{\mathbb{Q}}[e^{-r\tau_{B-x}} H(S_{\tau_{B-x}}) \mid S_0 = s,\ V_0 = v], \tag{3.6}$$

where τ_{B-x} and τ are defined in (1.1) and (2.4), respectively. Equivalently, the initial arbitrage-free price of the timer put option satisfies the following risk-neutral representation and conditional Black–Scholes–Merton-type formula

$$P_0 = \mathbb{E}^{\mathbb{Q}}[e^{-r\tau} \max(K - S_\tau, 0)] \tag{3.7a}$$

$$= \mathbb{E}^{\mathbb{Q}}[Ke^{-r\tau} N(-d_2(V_\tau, \tau)) - S_0(1 - e^{d_0(V_\tau,\tau)} N(d_1(V_\tau, \tau)))]; \tag{3.7b}$$

similar expressions follow for the time call option, i.e.,

$$C_0 = \mathbb{E}^{\mathbb{Q}}[e^{-r\tau} \max(S_\tau - K, 0)] \tag{3.8a}$$

$$= \mathbb{E}^{\mathbb{Q}}\big[S_0 e^{d_0(V_\tau,\tau)} N(d_1(V_\tau, \tau)) - Ke^{-r\tau} N(d_2(V_\tau, \tau))\big]. \tag{3.8b}$$

Proof. See Appendix A. □

REMARK 3.2. Based on (3.5), it follows that $\partial u/\partial t = 0$ for the PDE boundary value problem (3.3a) and (3.3b). Denote by $w(s, v, x) = u(t, s, v, x)$. Thus, we characterize the arbitrage-free timer option prices using the following Dirichlet problems of degenerated elliptic PDEs with boundary condition on a plane $\{(\xi_1, \xi_2, B), \xi_1 \in \mathbb{R}, \xi_2 \in \mathbb{R}\}$:

$$v\frac{\partial w}{\partial x} + \frac{1}{2}\sigma_v^2 v\frac{\partial^2 w}{\partial v^2} + \frac{1}{2}s^2 v\frac{\partial^2 w}{\partial s^2} + \rho\sigma_v s v\frac{\partial^2 w}{\partial s \partial v} + \kappa(\theta - v)\frac{\partial w}{\partial v} + rs\frac{\partial w}{\partial s} - rw = 0,$$
$$w(s, v, B) = H(s).$$

Indeed, this characterization reconciles the fact that timer option prices are independent of the initial time, but solely depend on the initial asset price, the variance level, and the variance budget. We also note that the remaining variance budget x can be regarded as a temporal variable corresponding to the stochastic variance clock (2.3).

An immediate mathematical reconcilement with the Black–Scholes–Merton formulas resides in the case of $\rho = 0, \sigma_v = 0, \kappa = 0$, under which $\{W_t^{(2)}\}$ is the only driving Brownian motion. In this case, the variance $V_t \equiv V_0$ is a constant and

$$dS_t = rS_t dt + \sqrt{V_0} S_t dW_t^{(2)}.$$

For a variance budget $B = V_0 T$, it is obvious that $\tau = T$. Thus,

$$S_\tau \equiv S_T = S_0 \exp\left\{rT - \frac{1}{2}B + \sqrt{B}Z\right\}.$$

It is obvious that $d_0(v, \xi)$ defined in (3.4a) equals zero; $d_1(v, \xi)$ and $d_2(v, \xi)$ defined in (3.4b) and (3.4c), respectively, reduce to the Black–Scholes–Merton case, i.e.,

$$d_1(v, \xi) = \frac{1}{\sqrt{V_0 T}}\left[\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}V_0\right)T\right],$$
$$d_2(v, \xi) = \frac{1}{\sqrt{V_0 T}}\left[\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}V_0\right)T\right].$$

The price of the timer call option with variance budget $B = V_0 T$ coincides with the Black–Scholes–Merton price of a call option with maturity T and strike K. That is, $C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$.

4. BESSEL PROCESSES, STOCHASTIC VOLATILITY, AND FIRST-PASSAGE TIME FOR REALIZED VARIANCE

Motivated by the valuation of timer options, we present a characterization of the joint distribution of the first-passage time τ defined in (2.4) and the variance V_τ via a Bessel process with constant drift. Also, a related joint distribution on Bessel processes is investigated.

First, we introduce some notations and briefly recall some fundamental facts about Bessel processes and Bessel processes with constant drift. According to Linetsky (2004), a Bessel process with drift μ and index $\nu > -\frac{1}{2}$, or equivalently, dimension $\delta = 2(\nu + 1) > 1$, has a state space $E = [0, \infty)$, a scale function $s(x) = x^{-2\nu}e^{-2\mu x}$, and a speed measure $m(dx) = 2x^{2\nu+1}e^{2\mu x}dx$. Its infinitesimal generator is given by

$$\mathcal{A}f(x) = \frac{1}{2}f''(x) + \left(\left(\nu + \frac{1}{2}\right)\frac{1}{x} + \mu\right)f'(x), \tag{4.1}$$

with the corresponding domain

$$\mathbb{D}_A = \left\{ f \mid f, \mathcal{A}(f) \in C_0([0, +\infty)), \frac{df^+(0)}{ds} = 0 \right\}, \tag{4.2}$$

where the reflecting boundary condition is defined by

$$\frac{df^+(0)}{ds} = \lim_{x \to 0+} \frac{f(x) - f(0)}{s(x) - s(0)} = 0. \tag{4.3}$$

For $\mu = 0$, the case reduces to a standard Bessel process, see Revuz and Yor (1999) for detailed discussions.

Let $\{R_t\}$ be a δ-dimensional Bessel process labeled as BES^δ. For any $\delta \geq 2$, $\{R_t\}$ is a diffusion process governed by the SDE:

$$dR_t = \frac{\delta - 1}{2R_t}dt + d\mathcal{W}_t, \quad R_0 > 0, \tag{4.4}$$

where $\{W_t\}$ is a standard Brownian motion. Conventionally, BES^δ is alternatively denoted by $BES^{(\nu)}$, where $\nu = \delta/2 - 1$ is its index. For any $\mu \in \mathbb{R}$, let $\{R_t^\mu\}$ be a δ-dimensional Bessel process with drift μ. For any $\delta \geq 2$, $\{R_t^\mu\}$ is a diffusion process $\{R_t^\mu\}$ governed by the SDE:

$$dR_t^\mu = \left(\frac{\delta - 1}{2R_t^\mu} + \mu\right)dt + d\mathcal{W}_t, \quad R_0^\mu > 0. \tag{4.5}$$

Similar to the Bessel process without drift, let BES_μ^δ or $BES_\mu^{(\nu)}$ denote such a process, where $\nu = \delta/2 - 1$ is its index. For the case of $\delta \geq 2$ $(\nu \geq 0)$, the point zero is unattainable for both $\{R_t\}$ are $\{R_t^\mu\}$; however, for the case of $2 > \delta \geq 1$ $(0 > \nu \geq -1/2)$, the point zero is attainable but instantaneously reflecting for both $\{R_t\}$ are $\{R_t^\mu\}$ (see Revuz and Yor 1999 and Linetsky 2004). According to exercise 1.26 in chapter XI from Revuz and Yor (1999), it is known that (4.4) and (4.5) still hold for $2 > \delta \geq 1$; however, these equations are regarded as semimartingale decompositions, which imply the following integrability properties:

$$\int_0^t \frac{1}{R_s}ds < \infty \quad \text{and} \quad \int_0^t \frac{1}{R_s^\mu}ds < \infty. \tag{4.6}$$

In queuing theory, Bessel processes with constant drift have appeared as heavy traffic limits (see Coffman, Puhalskii, and Reiman 1998); in financial engineering, Bessel processes with constant drift relate to some nonaffine analytically tractable specifications for spot interest rates, credit spreads, and stochastic volatility (see Linetsky 2004). For more detailed studies on Bessel processes and Bessel processes with drift, readers are referred to Revuz and Yor (1999), Karatzas and Shreve (1991) as well as Linetsky (2004).

4.1. A Distributional Identity

Based on (3.7b) and (3.8b), the valuation of timer options calls for the joint distribution of (V_τ, τ). It turns out that for the Feller diffusion

$$dV_t = \kappa(\theta - V_t)\,dt + \sigma_v\sqrt{V_t}dW_t^{(1)} \tag{4.7}$$

the distribution of (V_τ, τ) can be characterized by using a Bessel process with constant drift.

PROPOSITION 4.1. *Under the risk neutral probability measure $\mathbb{Q}$, we have a distributional identity for the bivariate random variable (V_τ, τ):*

$$(V_\tau, \tau) \overset{\mathcal{D}}{=} \left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right), \tag{4.8}$$

where τ and B are defined in (2.4) and $\{V_t\}$ is defined in (3.1b). Here $\{X_t\}$ is a Bessel process with index $\nu = \kappa\theta/\sigma_v^2 - 1/2$ (dimension $\delta = 2\kappa\theta/\sigma_v^2 + 1$) and constant drift $-\kappa/\sigma_v$ satisfying the stochastic differential equation:

$$dX_t = \left(\frac{\kappa\theta}{\sigma_v^2 X_t} - \frac{\kappa}{\sigma_v}\right) dt + d\mathcal{B}_t, \quad X_0 = V_0/\sigma_v, \tag{4.9}$$

where $\{\mathcal{B}_t\}$ is a standard one-dimensional Brownian motion.

Proof. See Appendix B. □

This proposition can be regarded as parallel to a characterization investigated in Geman and Yor (1993) for the Hull and White (1987) stochastic volatility model. Indeed, when the Feller condition $2\kappa\theta - \sigma_v^2 \geq 0$ holds, the dimension (resp. the index) of Bessel process with drift (4.9) satisfies $\nu = \kappa\theta/\sigma_v^2 - 1/2 \geq 0$ (resp. $\delta = 2\kappa\theta/\sigma_v^2 + 1 \geq 2$). Thus, zero is an unattainable point for both the variance process $\{V_t\}$ and the Bessel process $\{X_t\}$ as introduced in (3.1b) and (4.9), respectively, according to the Feller test for classifying boundary points (see Section 5.5 in Karatzas and Shreve 1991). Thus, in the case of $2\kappa\theta - \sigma_v^2 \geq 0$, the identity (4.8) follows from a stochastic time-change argument. When the Feller condition is violated, i.e., $2\kappa\theta - \sigma_v^2 < 0$, zero is attainable and instantaneously reflecting. The distributional identity (4.8) follows from the general theory of Markov diffusion processes and machineries from real analysis.

4.2. A Joint Density on Bessel Process

For the Bessel process $\{R_t\}$ with index $\nu \geq 0$, the joint distribution of $(R_t, \int_0^t \frac{1}{R_u^2}du)$ and its applications in applied probability and stochastic modeling are well studied in, e.g., Geman and Yor (1993), Revuz and Yor (1999), and Yor (2001). However, the law of $(R_t, \int_0^t \frac{1}{R_u}du)$ and its applications received much less attention. For any arbitrary $s > 0$, let

$$p(x, t; s)\,dxdt := \mathbb{P}\left(R_s \in dx, \int_0^s \frac{du}{R_u} \in dt\right). \tag{4.10}$$

To further apply the joint distribution in (4.8) for the analytical valuation of timer options, we find an explicit expression for $p(x, t; s)$ in what follows.

Now, we derive an expression for (4.10) by employing Laplace transform inversion on a joint density on Bessel processes involving exponential stopping given in Borodin and Salminen (2002). According to Borodin and Salminen (2002), the joint distribution on Bessel process and the integration functional of its reciprocal stopped at an independent exponential time admits the following closed-form representation; i.e., for the Bessel process R with index ν starting at $R_0 > 0$ and an independent exponential time T with intensity $\lambda > 0$, we have that

$$\mathbb{P}\left(R_T \in dx, \int_0^T \frac{du}{R_u} \in dt\right) = q(x, t; \lambda)\, dx dt, \tag{4.11}$$

where

$$q(x, t; \lambda) := \frac{\lambda\sqrt{2\lambda}x^{\nu+1}}{R_0^\nu \sinh\left(t\sqrt{\frac{\lambda}{2}}\right)} \exp\left(-\frac{(R_0 + x)\sqrt{2\lambda}\cosh\left(t\sqrt{\frac{\lambda}{2}}\right)}{\sinh\left(t\sqrt{\frac{\lambda}{2}}\right)}\right) I_{2\nu}\left(\frac{2\sqrt{2\lambda R_0 x}}{\sinh\left(t\sqrt{\frac{\lambda}{2}}\right)}\right)$$

with $I_\nu(\cdot)$ representing a modified Bessel function for the first kind defined by

$$I_\nu(z) := \sum_{k=0}^{+\infty} \frac{1}{k!\Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{\nu+k}. \tag{4.12}$$

We note that the exponential stopping (4.11) is equivalent to a Laplace transform of $p(x, t; s)$ on the time variable s, i.e.,

$$\mathcal{H}(\lambda) := \int_0^{+\infty} e^{-\lambda s} p(x, t; s)\, ds \equiv \frac{q(x, t; \lambda)}{\lambda}, \tag{4.13}$$

holding for all positive real values: $\lambda > 0$. To obtain $p(x, t; s)$ through Laplace transform inversion, we perform analytical continuation in order to extend the domain of the Laplace transform $\mathcal{H}(\lambda)$ to the following complex region of convergence (see, e.g., chapter 3 in Doetsch 1974):

$$\mathfrak{D} = \{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0\}. \tag{4.14}$$

In what follows, we articulate the analytical continuation of $q(x, t; \lambda)/\lambda$, i.e., we consider how this function is defined for $\lambda = \gamma + iy$ with $\gamma > 0$ and $i = \sqrt{-1}$. First, we note that there is no problem to define $\sqrt{\lambda} = \sqrt{\gamma + iy}$ as a single valued analytic function because $\gamma + iy$ lies in the principal branch $(-\pi, \pi]$ as y increases from $-\infty$ to ∞. Denoted by

$$A(y) := \frac{2\sqrt{2(\gamma + iy)R_0 x}}{\sinh\left(t\sqrt{\frac{\gamma+iy}{2}}\right)}. \tag{4.15}$$

In Figure 4.1, we observe the winding of $A(y)$ through the winding of a scaled argument

$$\gamma(y) := A(y)\frac{\log\log|A(y)|}{|A(y)|}.$$

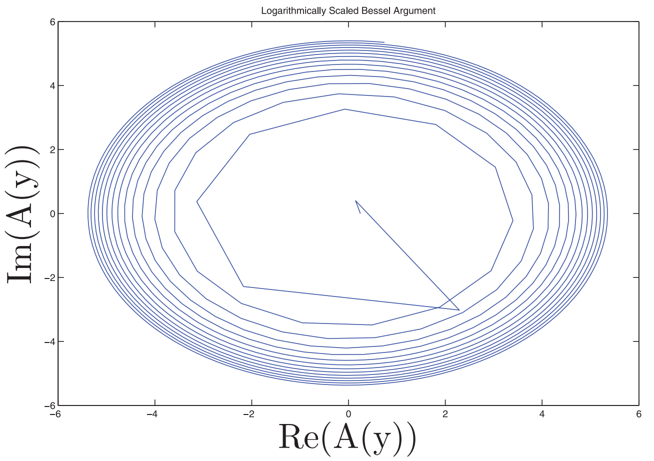


FIGURE 4.1. Winding of the Bessel argument $A(y)$.

We also note that brute force extension of the modified Bessel function (4.12) results in a multivalued function when the index ν is not an integer.[1] Thus, to ensure the analyticity of $I_{2\nu}(A(y))$ on the region $\mathfrak{D}$, we employ the following well-known analytical continuation, i.e., see page 376 in Abramowitz and Stegun (1984), for any $m \in \mathbb{N}$,

$$I_\nu(ze^{m\pi i}) = e^{m\nu\pi i} I_\nu(z). \tag{4.16}$$

Thus, to preserve analyticity, the value of the function $q(x,t;\lambda)/\lambda$ on a branch different from the principal one $(-\pi, \pi]$ needs to be defined by multiplying the value on the principal branch by a factor $e^{m\nu\pi i}$.

It is obvious that the Laplace transform is absolutely convergent in the region (4.14) in the sense that

$$\int_0^{+\infty} |e^{-\lambda s} p(x,t;s)|\, ds < \infty, \quad \text{for } \lambda \in \mathfrak{D}.$$

Note that Laplace transform is analytic in the region of absolute convergence, see, e.g., chapter 6 in Doetsch (1974). Therefore, the uniqueness of analytical continuation (see, e.g., chapter 8 in Ahlfors 1979) guarantees that (4.13) holds for any $\lambda \in \mathfrak{D}$ with the function $\mathcal{H}(\lambda)$ defined on $\mathfrak{D}$ through the aforementioned analytical continuation procedure. Similar argument for analytical continuation of Laplace transform applied to mathematical finance can be found in appendix D of Davydov and Linetsky (2001b). Thus, according to section 2.4 in Doetsch (1974) (see, e.g., theorems 24.3 and 24.4), the joint density (4.10) can be obtained from a Bromwich integral for inverting Laplace transform, of which the literature has seen various applications, see, e.g., Davydov and Linetsky (2001a), Davydov and Linetsky (2001b), Petrella (2004), Fusai (2004), Kou, Petrella, and Wang (2005), Cai, Chen, and Wan (2009, 2010), and Cai and Kou (2011, 2012).

[1] This is because the complex power function is multivalued. Indeed, the power function $z^\alpha := |z|^\alpha e^{i(arg(z)+2n\pi)\alpha}$, for any integer n, has different values when α is not an integer.

PROPOSITION 4.2. *The joint density (4.10) admits the following analytical representation through Laplace transform inversion: for any damping factor $\gamma > 0$, we have*

$$p(x, t; B) = \lim_{z\to\infty} \frac{1}{2\pi i} \int_{\gamma - iz}^{\gamma + iz} e^{B\lambda} \mathcal{H}(\lambda)\, d\lambda, \tag{4.17}$$

where the function $\mathcal{H}(\lambda)$ is defined in (4.13) and the above procedure of analytical continuation.

Before closing this section, we bring out the joint law of a standard Bessel process satisfying the stochastic differential equation (4.4) and its driving Brownian motion as a by-product of the previous results.

COROLLARY 4.3. *The joint density of R_t and $\mathcal{W}_t$ in (4.4) has the following representation*

$$\mathbb{P}(R_t \in dx, \mathcal{W}_t \in dw) = \frac{2\nu + 1}{2} p\left(x, \frac{2(x - w - R_0)}{2\nu + 1}, t\right) dxdw.$$

5. BLACK–SCHOLES–MERTON-TYPE FORMULAS FOR TIMER OPTION VALUATION

In this section, we apply the results established in the previous sections to derive Black–Scholes–Merton-type formulas for pricing timer options. Such formulas can be regarded as generalizations of the celebrated Black–Scholes–Merton formula (see Black and Scholes 1973 and Merton 1973) as well as the semi-closed-form formula under the Heston stochastic volatility model (see Heston 1993) for pricing European options. Under an assumption that the interest rate is zero, our formulas can be simplified to the Black–Scholes–Merton formula with appropriate parameters, based on which we point out some connections with existing literatures and untangle a puzzle on the comparison between timer options and European options.

Now, using $d_0(v, \xi)$, $d_1(v, \xi)$, and $d_2(v, \xi)$ defined in (3.4a), (3.4b), and (3.4c), the joint density $p(x, t)$ explicitly obtained in Proposition 4.2, and an auxiliary function defined by

$$c(v, \xi) := \frac{\kappa}{\sigma_v^2}(V_0 - v) + \frac{\kappa^2\theta}{\sigma_v^2}\xi - \frac{\kappa^2}{2\sigma_v^2} B,$$

we obtain the following proposition.

PROPOSITION 5.1. *Under the Heston (1993) stochastic volatility model (3.1a) and (3.1b), for strike K and variance budget B, the initial price of a timer call option is given by*

$$C_0 := S_0 \Pi_1^c - K \Pi_2^c; \tag{5.1}$$

the initial price of a timer put option is given by

$$P_0 := K \Pi_2^p - S_0 \Pi_1^p. \tag{5.2}$$

Here, for $i = 1,\ 2$,

$$\Pi_i^c := \int_0^\infty \int_0^\infty \Omega_i^c\left(\sigma_v x, \frac{t}{\sigma_v}\right) p(x, t; B)\, dx dt, \tag{5.3}$$

$$\Pi_i^p := \int_0^\infty \int_0^\infty \Omega_i^p\left(\sigma_v x, \frac{t}{\sigma_v}\right) p(x, t; B)\, dx dt,$$

where

$$\begin{aligned}
\Omega_1^c(v, \xi) &= N(d_1(v, \xi)) \exp\{d_0(v, \xi) + c(v, \xi)\},\\
\Omega_2^c(v, \xi) &= N(d_2(v, \xi)) \exp\{-r\xi + c(v, \xi)\},\\
\Omega_1^p(v, \xi) &= (1 - N(d_1(v, \xi)) \exp\{d_0(v, \xi)\}) \exp\{c(v, \xi)\},\\
\Omega_2^p(v, \xi) &= N(-d_2(v, \xi)) \exp\{-r\xi + c(v, \xi)\},
\end{aligned}$$

and $p(x, t; B)$, *as given in (4.17), is the transition density of a standard Bessel process with index* $\nu = \kappa\theta/\sigma_v^2 - 1/2$ *and initial value* $R_0 = V_0/\sigma_v$.

Proof. See Appendix C. □

An idea similar to timer options can be traced back to Bick (1995), which proposed a quadratic variation based and model-free portfolio insurance strategy to synthesize a put-like protection with payoff $\max\{K'e^{r\tau} - S_\tau, 0\}$ for some $K' > 0$. Though the timer option payoff $\max\{K' - S_\tau, 0\}$, for some $K' > 0$, considered in this paper is different from the put-like protection, timer put options may serve as effective tools for portfolio insurance. With a timer put option written on an index (a well-diversified portfolio), the uncertainty about the index's volatility is replaced by the variability in time horizon. Dupire (2005) applied a similar idea to the "business time delta hedging" of volatility derivatives under the assumption that the interest rate is zero. Working under a general semimartingale framework, Carr and Lee (2010) investigated the hedging of options on realized variance. As an example, Carr and Lee (2010) provided a model-free strategy for replicating a class of claims on asset price when realized variance reaches a barrier. Using the method proposed in Carr and Lee (2010), one is able to price and replicate a payoff in the form: e.g., $\max(S_\tau - Ke^{r\tau}, 0)$. It is worth noting that this payoff coincides with Société Générale's timer call option with payoff $\max(S_\tau - K, 0)$, when the interest rate r is assumed to be zero.

When $r = 0\%$, a much simpler version of the Black–Scholes–Merton-type formulas (5.1) and (5.2) for pricing timer options can be directly derived from the risk-neutral representations (3.7a) and (3.8a). Indeed, we have that

$$S_t = S_0 \exp\left\{\int_0^t \sqrt{V_u}\, dW_u^s - \frac{1}{2}\int_0^t V_u du\right\},$$

where $W_u^s = \rho W_t^{(1)} + \sqrt{1-\rho^2}\, W_t^{(2)}$. Recall that the variance budget is calculated as $B = \sigma_0^2 T_0$, where $[0,\ T_0]$ is the expected investment horizon for some T_0; and σ_0 is the forecasted annualized realized volatility. Based on the definition of τ in (2.4), we apply the *Dubins–Dambis–Schwarz* theorem (see, e.g., chapter 3, theorem 4.6 in Karatzas and Shreve 1991) to obtain that

$$\int_0^\tau \sqrt{V_u}\, dW_u^s = \mathcal{W}_B^s,$$

for a standard Brownian motion $\{\mathcal{W}_t^s\}$. This leads to that

$$S_\tau = S_0 \exp\left\{\mathcal{W}_B^s - \frac{1}{2}B\right\}.$$

It is easy to obtain the following result for pricing a timer call option under the assumption $r = 0\%$.

PROPOSITION 5.2. *Assuming $r = 0\%$, the price of the timer call option with strike K and variance budget $B = \sigma_0^2 T_0$ can be expressed by the Black–Scholes–Merton (1973) formula:*

$$C_0 = \mathbb{E}^{\mathbb{Q}}[\max\{S_\tau - K, 0\}] = BSM(S_0, K, T_0, \sigma_0, 0), \tag{5.4}$$

where $BSM(s, K, T, \sigma, r)$ is the Black–Scholes–Merton formula for pricing European call options, i.e.,

$$BSM(s, K, T, \sigma, r) := sN(d_1) - Ke^{-rT}N(d_2),$$

with the functions

$$d_1 := \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{s}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right],$$
$$d_2 := \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{s}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right].$$

Based on this proposition, we provide a theoretical justification of the following claim given in Sawyer (2007), i.e., "High implied volatility means call options are often overpriced. In the timer option, the investor only pays the real cost of the call and does not suffer from high implied volatility." More precisely, we verify that the timer call option with strike K and expected investment horizon T_0 and forecasted realized volatility σ_0 (variance budget $B = \sigma_0^2 T_0$) is less expensive than a European call option with strike K and maturity T_0, when the implied volatility $\sigma_{\text{imp}}(K, T_0)$ associated to strike K and maturity T_0 is higher than the realized volatility σ_0. Indeed, by (5.4), we deduce that

$$\mathbb{E}^{\mathbb{Q}}[\max\{S_\tau - K, 0\}] = BSM(S_0, K, T_0, \sigma_0, 0) \leq BSM(S_0, K, T_0, \sigma_{\text{imp}}(K, T_0), 0).$$

Comparing with European put options, timer put options are able to offer relatively cheaper cost of portfolio insurance and protection. If the realized variance is low, the timer put options take a long time to mature. Comparing with the regularly rolled European put options for protecting the downside risk of a portfolio, the timer put options require less frequency of rolling, resulting in a reduction in the cost for implementing the protection.

The comparison between timer options and European options heuristically motivates an option strategy for investors to capture the spread between the realized and implied volatility risk. For example, if an investor strongly believes that the current implied volatility is higher than the realized volatility over a certain period, she would take a short position in a European call option with maturity T_0, strike K, and implied volatility $\sigma_{\text{imp}}(K, T_0)$; and take a long position in a timer call option with the same strike K and variance budget $B = \sigma_{\text{imp}}(K, T_0)^2 T_0$. By the above analysis, the net value of this portfolio is zero at time zero. However, the timer call option has a maturity larger than T_0. At the time T_0, a positive profit is realized due to the premia of the timer option.

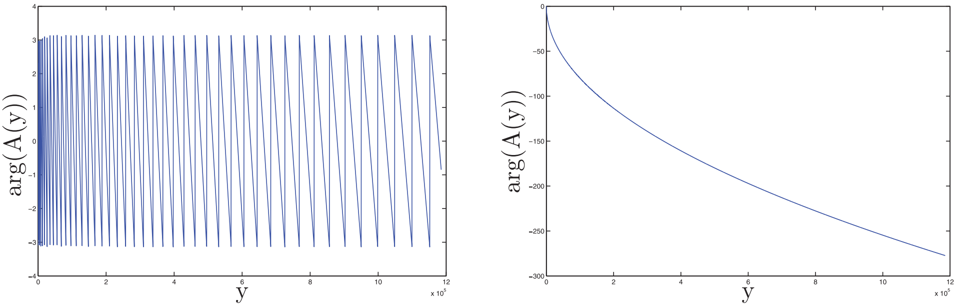
6. NUMERICAL IMPLEMENTATION

In this section, we briefly discuss the implementation of the analytical formulas proposed in theorem 4. An ad hoc Monte Carlo simulation scheme is also proposed in order to provide numerical benchmarks. Comparing with the Monte Carlo discretization approach, the analytical formula is bias-free since the numerical errors are entirely from Laplace transform inversion and numerical integration.

The implementation of the Black–Scholes–Merton-type formulas given in theorem 4 mainly consists of the following steps. To begin, we map the infinite integration domain to a finite rectangular domain $[0, 1] \times [0, 1]$ via a transform according to $u = e^{-x}$ and $z = e^{-t}$. Then, the two-dimensional integration on $[0, 1] \times [0, 1]$ converted from (5.3) can be implemented via the trapezoidal rule. Thus, the key task is to efficiently evaluate $p(-\log u, -\log z; B)$ at each grid point on $[0, 1] \times [0, 1]$. The implementation of the joint density $p(x, t; B)$ proposed in Proposition 4.2 requires correct valuation of special functions and numerical inversion of Laplace transforms.

When the inverse Laplace transform in (4.17) is implemented, the analytical continuation discussed in Subsection 4.2 needs to be taken into account. In practice, the winding of $A(y)$ defined in (4.15) must be captured to ensure the analyticity of the transformed function $\mathcal{H}(\lambda)$ in (4.17). However, most computation packages automatically map the complex numbers into the principal branch $(-\pi, \pi]$. This fact might cause the discontinuity of the Bessel function when its argument $A(y)$ goes across the negative real line. Therefore, an algorithm needs to be implemented to keep track of the winding number of the argument $A(y)$ by counting rotations and performing the analytical continuation via (4.16). This allows us to simply calculate the function on the principal branch and multiply it by a factor $e^{m\nu\pi i}$. A similar type of analytical correction can be found in Broadie and Kaya (2006) for exact simulation for the Heston stochastic volatility model. Figures 6.1(a) and (b) show the effect of rotation counting on the phase angle of $A(y)$.

Based on the correct valuation of the Laplace transform through analytical continuation as discussed in Subsection 4.2, we obtain the joint density (4.17) via an algorithm for inverting Laplace transforms. Numerical valuation of Laplace transform inversions has become popular in option pricing, see, e.g., Davydov and Linetsky (2001a), Petrella (2004), Fusai (2004) , Kou et al. (2005), Feng and Lin (2011), Cai et al. (2009, 2010), and Cai and Kou (2011, 2012). In this numerical experiment, we employ a well-known and widely used algorithm for inverting Laplace transforms from Fourier series expansion



(a) Uncorrected Angle (b) Corrected Angle

FIGURE 6.1. Correction for valuation of the Bessel argument $A(y)$.

proposed by Abate and Whitt (1992). A proper way of discretization and truncation as well as a suitable choice of damping parameter γ are essential for the efficient implementation. According to Abate and Whitt (1992), the trapezoidal rule works well for oscillatory integrands, since errors tend to cancel with each other. According to Abate and Whitt (1995), the damping parameter γ is usually chosen as $\gamma = A/(2B)$, where $A = \delta \log 10$, in order to have at most $10^{-\delta}$ discretization error for some integer δ. In practice, Abate and Whitt (1992) suggests that the choice of $A = 18.4$ should produce stable and accurate results. According to Abate and Whitt (1992), an efficient Euler algorithm for approximating the inversion can be proposed as follows:

$$p(x, t; B) \approx \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(B),$$

where $$s_{n+k}(B) := \frac{e^{A/2}}{2B} \mathrm{Re}\left\{\mathcal{H}\left(\frac{A}{2B}\right)\right\} + \frac{e^{A/2}}{B} \sum_{j=1}^{n+k} (-1)^j \,\mathrm{Re}\left\{\mathcal{H}\left(\frac{A + 2j\pi i}{2B}\right)\right\},$$

for some integers m and n. Detailed analysis of this efficient Laplace transform inversion algorithm can be found in Abate and Whitt (1992, 1995).

To set up benchmark values for illustrating the accuracy of our implementation based on the analytical formulas, we propose a Monte Carlo simulation scheme as follows. Note that alternative simulation strategies for pricing timer options have been investigated in Bernard and Cui (2011) . Instead of using the discounted payoff $e^{-r\tau} \max\{S_\tau - K, 0\}$ as the estimator directly, the expression (3.8b) offers a conditional Monte Carlo simulation estimator, which leads to the enhancement of efficiency through variance reduction. On the time grids $t_i = i\Delta t$, for $i = 1, 2, \ldots$, the bivariate distribution (V_τ, τ) is approximated via a "time-checking" algorithm based on an exact simulation of the discretized sample path of $\{V_t\}$. According to Cox et al. (1985), it is known that the transition of the square root diffusion

$$dV_t = \kappa(\theta - V_t)\, dt + \sigma_v \sqrt{V_t} dW_t^{(1)}$$

follows a noncentral chi-squared distribution. More precisely, V_t given V_u for $0 < u < t$, up to a scale factor, is a noncentral chi-squared distribution, i.e.,

$$V_t = \frac{\sigma_v^2(1 - e^{-\kappa(t-u)})}{4\kappa} \chi_d'^2 \left(\frac{4\kappa e^{-\kappa(t-u)}}{\sigma_v^2(1 - e^{-\kappa(t-u)})} V_u\right),$$

where the degree of freedom is $d = 4\theta\kappa/\sigma_v^2$ and the noncentrality parameter is $\lambda = \frac{4\kappa e^{-\kappa(t-u)}}{\sigma_v^2(1-e^{-\kappa(t-u)})} V_u$. By approximating the total variance $\int_0^t V_s ds$ using a trapezoidal rule, i.e.,

$$\int_0^{j\Delta t} V_s ds \approx \Delta t \left[\frac{V_0 + V(j\Delta t)}{2} + \sum_{k=1}^{j-1} V(k\Delta t)\right],$$

we check the first time when the variance budget is exhausted by searching the first $j \in \mathbb{N}$ (denoted by $j_{\min}$) such that

$$\Delta t \left[\frac{V_0 + V(j\Delta t)}{2} + \sum_{k=1}^{j-1} V(k\Delta t)\right] \geq B.$$

TABLE 6.1
Timer Call Option Price: Analytical Valuation and Monte Carlo Simulation

	$B = 0.01$			$B = 0.04$		
	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
Analytical values	12.3978	5.1835	1.4619	20.7765	14.0857	8.9501
CPU times (seconds)	58.28	60.31	59.15	59.35	58.96	60.72
Simulated values	12.3994	5.1818	1.4626	20.7693	14.0922	8.9576
Simulated exercise times	0.7130	0.7130	0.7130	3.6287	3.6287	3.6287
Standard errors	0.0059	0.0044	0.0024	0.0118	0.0105	0.0088
CPU times (seconds)	1551.84	1553.83	1550.63	1977.35	1977.44	1977.95

Notes. The parameters are set similar to those employed in Heston (1993) as $S_0 = 100$, $\rho = -0.3$, $V_0 = 0.02$, $\kappa = 2$, $\theta = 0.01$, and $\sigma_v = 0.1$, where the Feller condition holds.

TABLE 6.2
Timer Call Option Price: Analytical Valuation and Monte Carlo Simulation

	$B = 0.045$			$B = 0.18$		
	$K = 90$	$K = 100$	$K = 110$	$K = 90$	$K = 100$	$K = 110$
Analytical values	17.1397	11.1517	6.8609	29.7440	24.6997	20.4382
CPU times (seconds)	59.38	57.92	56.59	58.13	58.62	57.33
Simulation values	17.1300	11.1606	6.8545	29.7317	24.6916	20.4203
Simulated exercise times	0.9660	0.9660	0.9660	2.9249	2.9249	2.9249
Standard errors	0.0118	0.0101	0.0082	0.0128	0.0121	0.0113
CPU times (seconds)	15137.3	15087.7	15079.1	75070.2	71875.2	72936.7

Notes. The parameters are set similar to those employed in Broadie and Kaya (2006) as $S_0 = 100$, $\rho = -0.3$, $V_0 = 0.09$, $\kappa = 2$,$\theta = 0.09$, and $\sigma_v = 1$, where the Feller condition is violated.

Thus, we obtain an approximation $(V_\tau, \tau) \approx (V_{j_{\min}\Delta t}, j_{\min}\Delta t)$. Finally, we evaluate the estimator as

$$\widetilde{C_\tau} = S_0 e^{d_0(V_{j_{\min}\Delta t}, j_{\min}\Delta t)} N(d_1(V_{j_{\min}\Delta t}, j_{\min}\Delta t)) - K e^{-r j_{\min}\Delta t} N(d_2(V_{j_{\min}\Delta t}, j_{\min}\Delta t)).$$

In the numerical experiments, we price timer call options using two sets of model parameters corresponding to the cases when the Feller condition ($2\kappa\theta - \sigma_v^2 \geq 0$) holds or not, for which numerical results are reported in Tables 6.1 and 6.2. For each parameter set, we consider both small and large variance budgets ($B = \sigma_0^2 T_0$) and representative strikes corresponding to different moneyness ($S_0 = 100$, $K = 90$, $K = 100$, and $K = 110$). In Table 6.1, we assume the expected investment horizon as $T_0 = 0.6$ (resp. $T_0 = 3.5$) and assume the forecasted volatility $\sigma_0 = 0.13$ (resp. $\sigma_0 = 0.11$). In Table 6.2, we assume the expected investment horizon as $T_0 = 0.96$ (resp. $T_0 = 2.9$) and assume the forecasted volatility $\sigma_0 = 0.22$ (resp. $\sigma_0 = 0.25$).

Analytical values are obtained from implementing the analytical formulas proposed in Proposition 5.1 by plugging in the joint density (4.17). For simulating timer option price, we imitate the asymptotically optimal rule proposed in Duffie and Glynn (1995) for allocating computational resources by specifying $k = [\sqrt{n}]$, where n denotes the total number of simulation trials and k denotes the expected number of time steps in the expected investment interval $[0, T_0]$. In numerical experiments, we choose the number of simulation trials n such that standard errors are at a magnitude of about 10^{-2} or less.

The algorithms are implemented in Mathematica and performed on a laptop PC with an Intel(R) Pentium(R) M 1.73 GHz processor and 2 GB of RAM running Windows XP Professional. The computing times for analytical calculation through our pricing formulas (5.1) and (5.2) are around 1 minute on average, see the CPU times (seconds) reported under the analytical values. Since the Abate–Whitt algorithm for Laplace transform inversion is notably efficient, most of the computing time is employed by correct evaluation of the Laplace transform function (4.13) using the aforementioned analytical continuation algorithm. However, for obtaining simulation results with standard errors at most 10^{-2} as those listed in Tables 6.1 and 6.2, the elapsed CPU times for simulating each price range from about a half to several hours. We note that all our analytical values are contained in the 95% confidence intervals constructed via the simulated values and ± 1.96 times of the corresponding standard errors. This demonstrates the accuracy and efficiency of our analytical implementation.

7. CONCLUDING REMARKS

Motivated by analytical valuation of timer options, we explore their novel mathematical connection with stochastic volatility and Bessel processes (with constant drift). Under the Heston (1993) stochastic volatility model, we formulate the problem through a first-passage time problem on realized variance and generalize the standard risk-neutral valuation theory for fixed maturity options to a case involving random maturity. By time change and the general theory of Markov diffusions, we characterize the joint distribution of the first-passage time for realized variance and the corresponding variance using Bessel processes with drift. Thus, explicit formulas for a useful joint density related to Bessel processes are derived via Laplace transform inversion. Based on these theoretical findings, we obtain a Black–Scholes–Merton-type formula for pricing timer options and thus extend the analytical tractability of the Heston model. Several issues regarding the numerical implementation are briefly discussed.

As for further research topics, it will be interesting to investigate the valuation of timer options under more sophisticated models such as jump-diffusion stochastic volatility models. Due to the uncertainty in the maturity, it is also interesting to take into account more risk factors, e.g., the interest rate and dividend. It is also worth exploring more properties for Bessel process (with drift) and their applications in various fields.

APPENDIX A: PROOF OF PROPOSITION 3.1

Proof. The proof of (3.5) and (3.6), and equivalently, (3.7a) and (3.8a), follows from a generalization of the standard argument for risk-neutral valuation to a case of random maturity. For the sake of space, I omit the detailed argument and focus on the derivation of the conditional Black–Scholes–Merton formulas (3.7b) and (3.8b) instead. Without loss of generality, we establish (3.7b) as follows.

We begin by representing the solution to the stochastic differential equations (3.1a) and (3.1b) as

$$S_t = S_0 \exp\left\{rt - \frac{1}{2}\int_0^t V_s ds + \rho \int_0^t \sqrt{V_s} dW_s^{(1)} + \sqrt{1-\rho^2}\int_0^t \sqrt{V_s} dW_s^{(2)}\right\},$$
$$V_t = V_0 + \kappa\theta t - \kappa \int_0^t V_s ds + \sigma_v \int_0^t \sqrt{V_s} dW_s^{(1)}.$$

Through straightforward algebraic computations and the definition of τ in (2.4), we obtain that

$$\begin{aligned} S_\tau &= S_0 \exp\left\{r\tau - \frac{1}{2}\int_0^\tau V_s ds + \frac{\rho}{\sigma_v}\left(V_\tau - V_0 - \kappa\theta\tau + \kappa\int_0^\tau V_s ds\right)\right. \\ &\qquad \left. + \sqrt{1-\rho^2}\int_0^\tau \sqrt{V_s} dW_s^{(2)}\right\} \\ &= S_0 \exp\left\{r\tau - \frac{1}{2}B + \frac{\rho}{\sigma_v}(V_\tau - V_0 - \kappa\theta\tau + \kappa B) + \sqrt{1-\rho^2}\int_0^\tau \sqrt{V_s} dW_s^{(2)}\right\}. \end{aligned}$$

Since the variance process $\{V_s\}$ is independent of Brownian motion $\{W_s^{(2)}\}$, an analogy to Example 4.7.3 in Shreve (2004) (see page 173) yields the following distributional identity:

$$\int_0^\tau \sqrt{V_s} dW_s^{(2)} \mid \mathcal{F}_\tau^V \underline{\underline{\mathcal{D}}}\, N\left(0, \int_0^\tau V_s ds\right) \mid \mathcal{F}_\tau^V \equiv N(0, B) \mid \mathcal{F}_\tau^V,$$

where $\{\mathcal{F}_t^V\}$ denotes the filtration generated by $\{V_s\}$. Because τ and V_τ are $\mathcal{F}_t^V$– measurable, we deduce that

$$\begin{aligned} P_0 &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\{K - S_\tau, 0\} \mid \mathcal{F}_\tau^V\right]\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\{K - S_0\exp\{p+qZ\}, 0\} \mid \mathcal{F}_\tau^V\right]\right], \end{aligned}$$

where Z is a standard normal variable independent of $\mathcal{F}_\tau^V$ and

$$p = r\tau - \frac{1}{2}B + \frac{\rho}{\sigma_v}(V_\tau - V_0 - \kappa\theta\tau + \kappa B) \quad \text{and } q = \sqrt{(1-\rho^2)B},$$

Thus, it follows that

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}\max\{K - S_0\exp\{p+qZ\}, 0\} \mid \mathcal{F}_\tau^V\right] \\ &= e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(K - S_0\exp\{p+qZ\})1_{\{S_0\exp\{p+qZ\}\le K\}} \mid V_\tau, \tau\right] \\ &= e^{-r\tau}K\mathbb{Q}\left(Z \le \frac{1}{q}\left(\log\frac{K}{S_0} - p\right)\middle| V_\tau, \tau\right) - e^{-r\tau}S_0 \\ &\quad \mathbb{E}^{Q}\left[\exp\{p+qZ\}1_{\left\{Z\le\frac{1}{q}\left(\log\frac{K}{S_0}-p\right)\right\}}\middle| V_\tau, \tau\right]. \end{aligned}$$

Hence, the representation (3.7b) for timer put options follows from straightforward calculations of the above two terms based on the standard normal distribution. □

APPENDIX B: PROOF OF PROPOSITION 4.1

Proof. Recall from the definition (1.1) that

$$\tau_t = \inf\left\{u \geq 0, \int_0^u V_s\,ds = t\right\}.$$

For the local martingale

$$M_t = \int_0^t \sqrt{V_s}\,dW_s^{(1)}, \tag{B.1}$$

the *Dubins–Dambis–Schwarz* theorem (see, e.g., chapter 3, theorem 4.6 in Karatzas and Shreve 1991) yields that

$$M(\tau_t) = \int_0^{\tau_t} \sqrt{V_s}\,dW_s^{(1)} = \mathfrak{B}_t, \tag{B.2}$$

where $\{\mathfrak{B}_t\}$ is a standard one-dimensional Brownian motion.

To begin, we prove the distributional identity (4.8) under the Feller condition $2\kappa\theta - \sigma_v^2 \geq 0$, under which $V_t > 0$ for all $t \geq 0$. Since $f(u) = \int_0^u V_s ds$ is an increasing differentiable function, we have that

$$\tau_t = \int_0^t \frac{1}{V_{\tau_s}} ds. \tag{B.3}$$

Owing to

$$V_{\tau_t} = V_0 + \int_0^{\tau_t} \kappa(\theta - V_s)\,ds + \sigma_v \int_0^{\tau_t} \sqrt{V_s}\,dW_s^{(1)},$$

it follows that

$$V_{\tau_t} = V_0 + \int_0^t \frac{1}{V_{\tau_s}} \kappa(\theta - V_{\tau_s})\,ds + \sigma_v \mathfrak{B}_t.$$

For $\mathfrak{X}_t := V_{\tau_t}/\sigma_v$, we have that

$$\mathfrak{X}_t = \frac{V_0}{\sigma_v} + \int_0^t \left(\frac{\kappa\theta}{\sigma_v^2 \mathfrak{X}_u} - \frac{\kappa}{\sigma_v}\right) du + \mathfrak{B}_t.$$

Observing that $2\kappa\theta/\sigma_v^2 + 1 \geq 2$, the uniqueness of the solution to the SDE (4.9) yields that

$$\mathfrak{X}_B \stackrel{\mathcal{D}}{=} X_B.$$

Thus, we obtain that

$$(V_\tau, \tau) \stackrel{\mathcal{D}}{=} \left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right).$$

For the case $2\kappa\theta - \sigma_v^2 < 0$ where the Feller condition is violated, zero is attainable but instantaneously reflecting for the process $\{V_t\}$. So, the state space of the Markov process $\{V_{\tau_t}\}$ is $[0, \infty)$. In what follows, we employ the general theory of Markov diffusion processes (see chapter VII in Revuz and Yor 1999) to establish the distributional identity (4.8). The proof is carried out in the following three major steps.

First, we find the infinitesimal generator for $\{V_{\tau_t}\}$. For the process $\{V_{\tau_t}\}$ starting at $V_{\tau_0} \equiv x > 0$, we introduce a first hitting time for zero as follows:

$$T_0 = \inf\{u > 0,\ V_{\tau_u} = 0\}.$$

For an arbitrary function $f \in \mathcal{C}^2$, let

$$M_t^f := f(V_{\tau_t}) - f(x) - \int_0^t (\mathcal{B}f)(V_{\tau_s})\, ds,$$

where the differential operator is defined by

$$\mathcal{B}f(x) = \frac{1}{2}\sigma_v^2 f''(x) + \left(\frac{\kappa\theta}{x} - \kappa\right) f'(x). \tag{B.4a}$$

We note that, for all $u < T_0$, (B.3) implies that

$$d\tau_u = \frac{du}{V_{\tau_u}}.$$

Thus, we deduce that

$$\begin{aligned}
M_{t\wedge T_0}^f &= f(V_{\tau_{t\wedge T_0}}) - f(x) - \int_0^{t\wedge T_0} (\mathcal{B}f)(V_{\tau_s}) V_{\tau_s} d\tau_s \\
&= f(V_{\tau_{t\wedge T_0}}) - f(x) - \int_0^{\tau_{t\wedge T_0}} (\mathcal{B}f)(V_u) V_u du \\
&= f(V_{\tau_{t\wedge T_0}}) - f(x) - \int_0^{\tau_{t\wedge T_0}} \left[\frac{1}{2}\sigma_v^2 V_u f''(V_u) + \kappa(\theta - V_u) f'(V_u)\right] du \\
&= \int_0^{\tau_{t\wedge T_0}} f'(V_u)\sigma_v \sqrt{V_u}\, dW_u^{(1)}.
\end{aligned}$$

Now, based on (B.1), we perform integration variable substitution via $u = \tau_r$ and apply the *Dubins–Dambis–Schwarz* theorem (see, e.g., chapter 3, theorem 4.6 in Karatzas and Shreve 1991) to deduce that

$$M_{t\wedge T_0}^f = \int_0^{\tau_{t\wedge T_0}} f'(V_u)\sigma_v dM_u = \int_0^{t\wedge T_0} f'(V_{\tau_r})\sigma_v dM_{\tau_r} = \int_0^{t\wedge T_0} f'(V_{\tau_r})\sigma_v d\mathfrak{B}_r.$$

Thus, $\{M_{t\wedge T_0}^f\}$ is a martingale. According to section VII.1 in Revuz and Yor (1999), we conclude that the infinitesimal generator of V_{τ_t} on $(0, \infty)$ is (B.4a).

Second, in order to prove the instantaneous reflecting property at zeros, we resort to the method for analyzing boundary behavior of Markov diffusion discussed in chapter VII of Revuz and Yor (1999). Let $m_\xi(A)$ denote the speed measure of a set A for a process $\{\xi_t\}$. Since zero is an instantaneously reflecting point for the variance process $\{V_t\}$, we have $m_V(\{0\}) = 0$. Thus, the zero set of $\{V_t\}$ has zero Lebesgue measure, i.e.,

$$\lambda\{t \geq 0 : V_t = 0\} = 0, \quad a.e.\ \Omega.$$

This implies that the zero set of $\{V_{\tau_t}\}$ has Lebesgue measure zero

$$\lambda\{t \geq 0 : V_{\tau_t} = 0\} = 0, \quad a.e.\Omega.$$

Thus, the speed measure satisfies $m_{V_\tau}(\{0\}) = 0$. According to proposition (3.13) in chapter VII of Revuz and Yor (1999), for any test function f in the domain for the infinitesimal generator of $\{V_{\tau_t}\}$, we have

$$\frac{df^+}{ds}(0) = m_{V_\tau}(\{0\})\mathcal{B}f(0) = 0.$$

Finally, the theory of general Markov diffusion process guarantees that V_{τ_t} is characterized by the infinitesimal generator

$$\mathcal{B}f(x) = \frac{1}{2}\sigma_v^2 f''(x) + \left(\frac{\kappa\theta}{x} - \kappa\right) f'(x),$$

with domain

$$\mathbb{D}_B = \left\{ f \middle| f, \mathcal{B}(f) \in C_0([0, +\infty)), \frac{df^+(0)}{ds} = 0 \right\},$$

where the reflection condition is given by (4.3). Hence, we showed that $\{V_{\tau_t}/\sigma_v\}$ is equivalent in distribution to a Bessel process $\{X_t\}$ with drift $\mu = -\kappa/\sigma_v$ and index $0 > \nu = \kappa\theta/\sigma_v^2 - 1/2 > -1/2$; i.e., $\{V_{\tau_t}\}\underline{\underline{\mathcal{D}}}\{\sigma_v X_t\}$.

Finally, we prove

$$\tau_t = \int_0^t \frac{ds}{V_{\tau_s}}.$$

For almost every $\omega \in \Omega$, we define

$$g(t, \omega) := \int_0^t V_s(\omega)\, ds.$$

It is obvious that g is continuously differentiable. Let

$$\mathcal{Z} := \{s > 0,\ V_s(\omega) = 0\}$$

be the zero set of $V(\omega)$. For any $t \in \mathcal{Z}^c$, the function g has a nonzero derivative $g'(t, \omega) = V_t(\omega)$. According to the inverse function theorem (see pp. 221–223 in Rudin 1976), g is invertible in a neighborhood of t ; the inverse g^{-1} is continuously differentiable and satisfies that

$$\left(g^{-1}\right)'(g(t, \omega), \omega) = \frac{1}{g'(t, \omega)}. \tag{B.5}$$

Based on the definition of function inverse, we have

$$g^{-1}(s, \omega) = \tau_s(\omega) = \inf\left\{ t \geq 0,\ \int_0^t V_u(\omega)\, du = s \right\}. \tag{B.6}$$

Thus, (B.5) implies that

$$\frac{d\tau_s}{ds}(\omega)|_{s=g(t,\omega)} = \frac{1}{g'(t, \omega)} = \frac{1}{V_t(\omega)}. \tag{B.7}$$

Since g is absolute continuous, the *Luzin* property (see Rudin 1976) guarantees that the Lebesgue measure for $g(Z, \omega)$ is zero. We also note that the process $\{V_t\}$ is ergodic (see discussions in Göing-Jaeschke and Yor 2003 and Alaya and Kebaier 2011) and proposition 4.2 in Alaya and Kebaier 2011 implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t V_s ds = \theta$$

almost surely. So, we have $\int_0^t V_s ds \to \infty$ as $t \to \infty$. Thus, the range of $g(t, \omega)$ is $\mathbb{R}^+$ for almost all $\omega \in \Omega$. Hence, for almost every $s \in \mathbb{R}^+$, there exists $t \in \mathcal{Z}^c$ such that $g(t, \omega) = s$. So, from (B.7), we have

$$\frac{d\tau_s}{ds}(\omega) = \frac{1}{V_{g^{-1}(s,\omega)}(\omega)} = \frac{1}{V_{\tau_s}(\omega)},$$

holding almost everywhere for $s > 0$. Therefore, integration on the both sides leads to

$$\tau_t(\omega) \equiv \tau_t(\omega) - \tau_0(\omega) = \int_0^t \frac{d\tau_s}{ds}(\omega) ds = \int_0^t \frac{1}{V_{\tau_s}(\omega)} ds \overset{\mathcal{D}}{=} \int_0^t \frac{1}{\sigma_v X_t(\omega)} ds.$$

□

APPENDIX C: PROOF OF PROPOSITION 5.1

Proof. Based on Proposition 3.1 and Proposition 4.1, we obtain that

$$\text{(C.1)} \qquad C_0 = \mathbb{E}^{\mathbb{Q}}\Bigg[S_0 e^{d_0\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)} N\left(d_1\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)\right) - K e^{-r\int_0^B \frac{ds}{\sigma_v X_s}} N\left(\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)\right)\Bigg],$$

and

$$\text{(C.2)} \qquad P_0 = \mathbb{E}^{\mathbb{Q}}\Bigg[K e^{-r\tau} N\left(-d_2\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)\right) - S_0\left(1 - e^{d_0\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)} N\left(d_1\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)\right)\right)\Bigg].$$

To apply the density given in Proposition 4.2, we change the probability measure $\mathbb{Q}$ to a new one under which $\{X_t\}$ is a standard Bessel process. Indeed, we let $\widehat{\mathcal{B}}_t = \mathcal{B}_t - \kappa t/\sigma_v$. By the Girsanov theorem, $\{\widehat{\mathcal{B}}_t\}$ is a standard Brownian motion under a new probability measure $\widehat{\mathbb{Q}}$ constructed through the Radon–Nikodym derivative

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}}\bigg|_{\mathcal{F}_t} = \exp\left\{ \frac{\kappa}{\sigma_v} \mathcal{B}_t - \frac{1}{2}\left(\frac{\kappa}{\sigma_v}\right)^2 t \right\}.$$

Thus, under the probability measure $\widehat{\mathbb{Q}}$, $\{X_t\}$ is a standard Bessel process satisfying the following semimartingale decomposition:

$$dX_t = \frac{\kappa\theta}{\sigma_v^2 X_t} dt + d\widehat{\mathcal{B}}_t, \quad X_0 = V_0/\sigma_v.$$

Some algebraic computation yields that

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \exp\left\{\frac{\kappa}{\sigma_v}\left(X_t - \frac{V_0}{\sigma_v}\right) - \frac{\kappa}{\sigma_v}\int_0^t \frac{\kappa\theta}{\sigma_v^2 X_s} ds + \frac{1}{2}\left(\frac{\kappa}{\sigma_v}\right)^2 t\right\}.$$

It follows from (C.1) that

$$C_0 = \mathbb{E}^{\widehat{\mathbb{Q}}}\Bigg[\Bigg[S_0 e^{d_0\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)} N\left(d_1\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)\right) \\ - K e^{-r\int_0^B \frac{ds}{\sigma_v X_s}} N\left(\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)\right)\Bigg]\frac{d\mathbb{Q}}{d\widehat{\mathbb{Q}}}\Big|_{\mathcal{F}_B}\Bigg].$$

Thus, the timer call option price admits the following representation:

$$C_0 = \mathbb{E}^{\mathbb{P}_0}\left[S_0\Omega_1^c\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right) - K\Omega_2^c\left(\sigma_v X_B, \int_0^B \frac{ds}{\sigma_v X_s}\right)\right], \tag{C.3}$$

where the functions $\Omega_1^c(v, \xi)$ and $\Omega_2^c(v, \xi)$ are defined in Proposition 4.2. Combining with the joint density given in Proposition 4.2, we establish the Black–Scholes–Merton-type formula (5.1) for pricing timer call options. The formula (5.2) for pricing timer put options can be similarly proved. □

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